A model problem for a weight optimization

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Abstract

A weight-optimization method for the uniaxial bar under compressive force and self-weight is proposed. The first (initial) step is the shape-finding analysis under the assumption of a uniform stress distribution along the height of the bar. The second step is the weight optimization, i.e. the determination of minimum cross section for the given initial shape, load and material. The weight of the bar is taken as the objective function, the maximum stress in the bar is the state variable, and a Least-Squares algorithm (LSQNONLIN) is the optimization algorithm. We found that a significant mass reduction is achieved with the proposed optimizer and that this method is applicable in the shape optimization when an initial surface is given. We anticipate our method to be a starting point for the optimization of more complex geometries.

1. Introduction

Optimization is a mathematical process that, under certain conditions, determines the maximum or minimum value of a specified function. In the ideal case, one would like to obtain the exact solution for the design situation under consideration. In reality, however, one can only achieve an approximate solution.

The quantities numerically calculated during the process of obtaining the optimal solution are called the design variables [1].

The best solution selection process is based on a criterion described by a so-called objective function, which depends on the design variables. The set of candidate solutions that belong to the domain of the objective function and satisfy the so-called problem's constraints (typically certain equalities or inequalities) is called the design space (or feasible region).

Many studies have been made on optimization problems treating the cases where geometric configurations of structures are to be specified and where only the dimensions of their elements, such as the area of cross sections, are to be determined in order to attain the minimum structural weight (or costs). Many methods have been developed for the determination of a local minimum for the optimization problem [2, 5, 6].

In this paper, the cross-section of a body of revolution is used as a design variable to determine its minimum mass under axial load.

2. Analytical solution

2.1 Description of the problem

We considered an axial bar, fixed on the bottom side and loaded with an uniaxial compressive force on the top side as shown in *Fig. 1*. In this example the constant compressive stress $\sigma = \sigma_0$ is given, the normal force F is also given, and the area *A is unknown*. The weight of the bar cannot be neglected. The goal is to determine a function that describes the shape of the bar $\varphi(x) = r(x)$, with the assumption of the uniform stress distribution when height, density and initial radius are given.

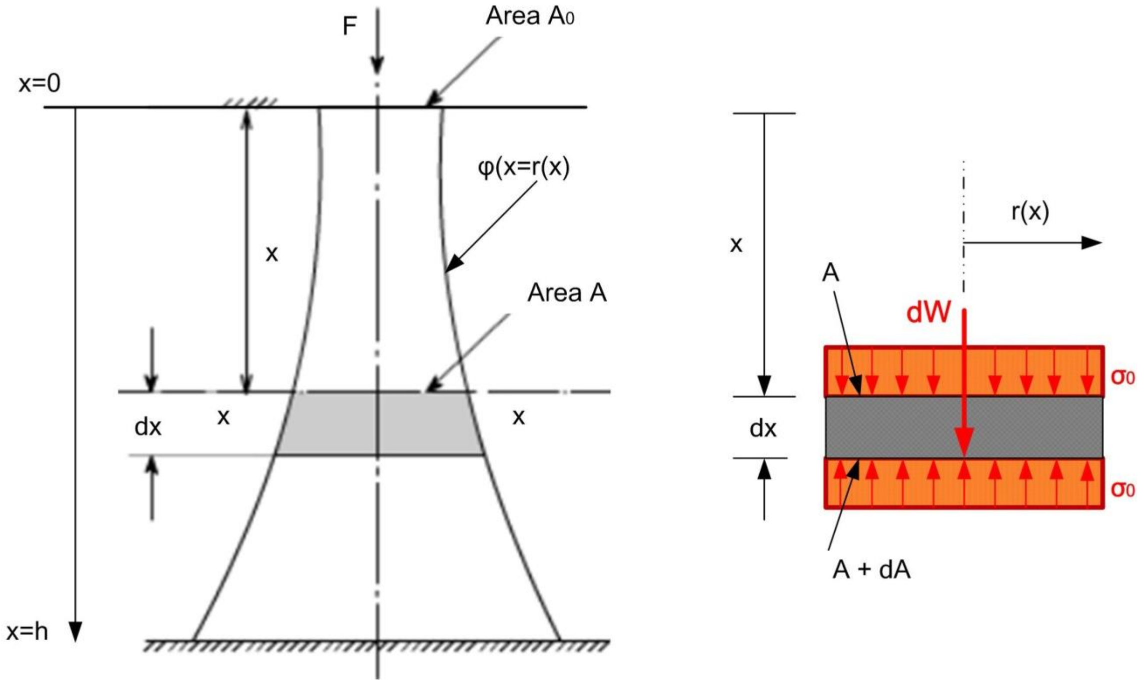

Fig. 1: Bar under compressive force [4]

2.2 Mathematical model

We introduce the coordinate *x* as shown in *Fig. 1* and consider a slice element of length d*x*. The circular cross-sectional area as a function of *x* is:

$$A(x) = r^2(x)\pi \tag{1}$$

where $r = r(x)$ is the unknown radius.

The normal force at the location x is given by $N = \sigma_0 \mathrm{A}$. At the location x+dx, the area and the normal force are $A + dA$ and $N + dN = \sigma_0(A + dA)$. The weight of the element is $dW = \rho g dV$ where $dV = Adx$. The weight of the element above the section xx is:

$$W = \int_0^x \rho g A dx \,. \tag{2}$$

The equilibrium condition in the vertical direction yields:

$$\uparrow \sigma_0(dA + A) - \rho g dV - \sigma_0 \mathrm{A} \rightarrow \sigma_0 A dx - \rho g A dx = 0. \tag{3}$$

Separation of variables and integration lead to:

$$\int \frac{dA}{A} = \int \frac{\rho g dx}{\sigma_0}. \tag{4}$$

$$log_e^A = \frac{\rho g x}{\sigma_0} + \mathrm{C}. \tag{5}$$

The constant of integration C is determined by applying the boundary conditions at $x = 0; A = A_0$ or $C = log_e^{A_0}$ which yields:

$$log_e^A = \frac{\rho g x}{\sigma_0} + log_e^{A_0}, \tag{6}$$

$$log_e\left(\frac{A}{A_0}\right) = \frac{\rho g x}{\sigma_0}, \tag{7}$$

$$\text{or} \quad e^{\frac{\rho g x}{\sigma_0}} = \frac{A}{A_0}\,. \tag{8}$$

Also we have for $x = 0 \rightarrow \sigma = \frac{F}{A_0}$, so eq. (8) becomes:

$$\frac{A}{A_0} = e^{\frac{\rho g x A_0}{F}}\,. \tag{9}$$

Thus:

$$A(x) = A_0 e^{\frac{\rho g x \pi A_0}{F}} = \pi r_0^2 e^{\frac{\rho g x \pi r_0^2}{F}} = \pi r^2(x). \tag{10}$$

The bar’s radius is:

$$\varphi(x) = r(x) = \sqrt{r_0^2 e^{\frac{\rho g x \pi r_0^2}{F}}}. \tag{11}$$

Inserting (10) into (2) we obtain the total weight of the bar:

$$W = F\left[\left(e^{\frac{\rho r_0^2 g h \pi}{F}}\right) - 1\right]. \tag{12}$$

3. Approximate solution with a polynomial generatrix

The generatrix φ that defines the shape of a bar with uniform stress distribution within the structure under the action of its own weight and a load F given by eq. (11) is computed in Matlab for the cylinder geometry and load as: F=0.001 [N], ρ=1.7 [kg/m^3], h=1 [m], $\mathrm{r}_0 = 0.01$ [m]. *Fig.2.* shows the Matlab’s plot of the eq. (11).

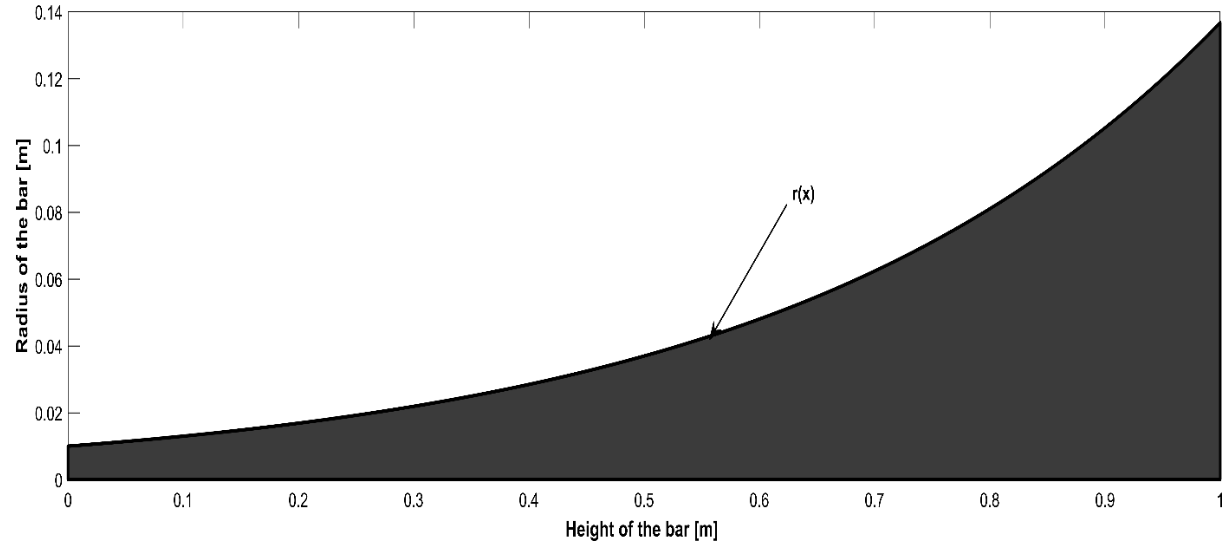

Fig.2: Radius of the bar r(x)

When r is calculated according to the eq. (11) and F tends to infinity (all other quantities being fixed), we get a result which is independent of F as shown in the *Fig.3*.

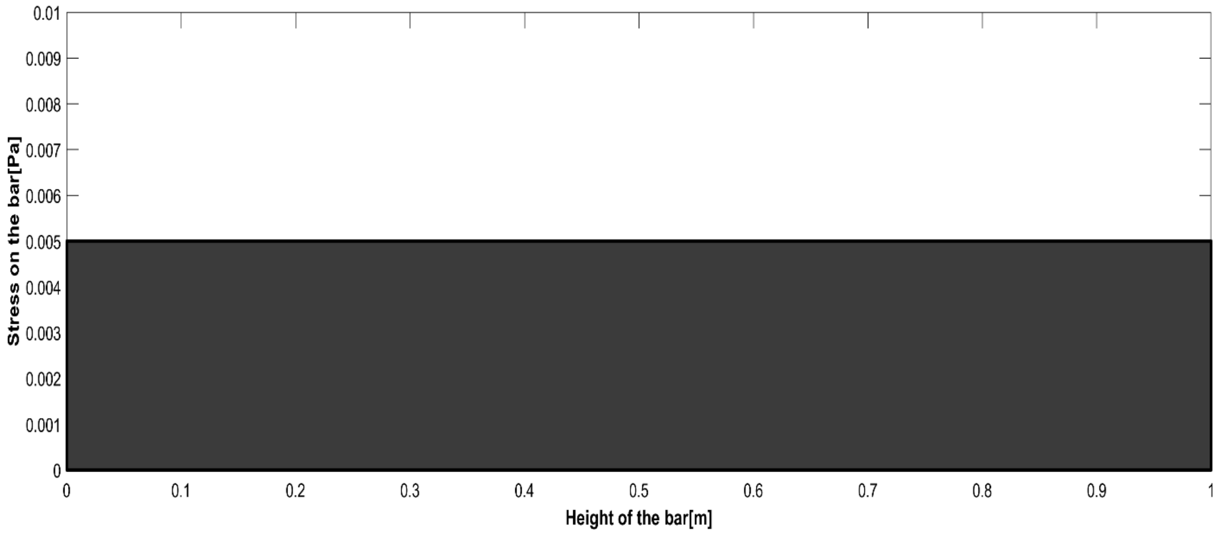

Fig. 3: Radius of the bar r(x) for the force with the magnitude of 35 [kN]

From the mathematical point of view, when F tends to infinity and all other values being fixed, then the weight part ρg of the eq. (11) does not have any impact on the final result. That is, the solution is again a circular cylinder which satisfies the assumption for a structure with a uniform stress distribution.

The normal stress is a consequence of the normal force and bar's own weight.When the weight is taken into consideration normal stress is defined as:

$$\sigma = \frac{F+W}{A}. \tag{13}$$

A stress profile in the structure is investigated and shown in the *Fig. 4.*

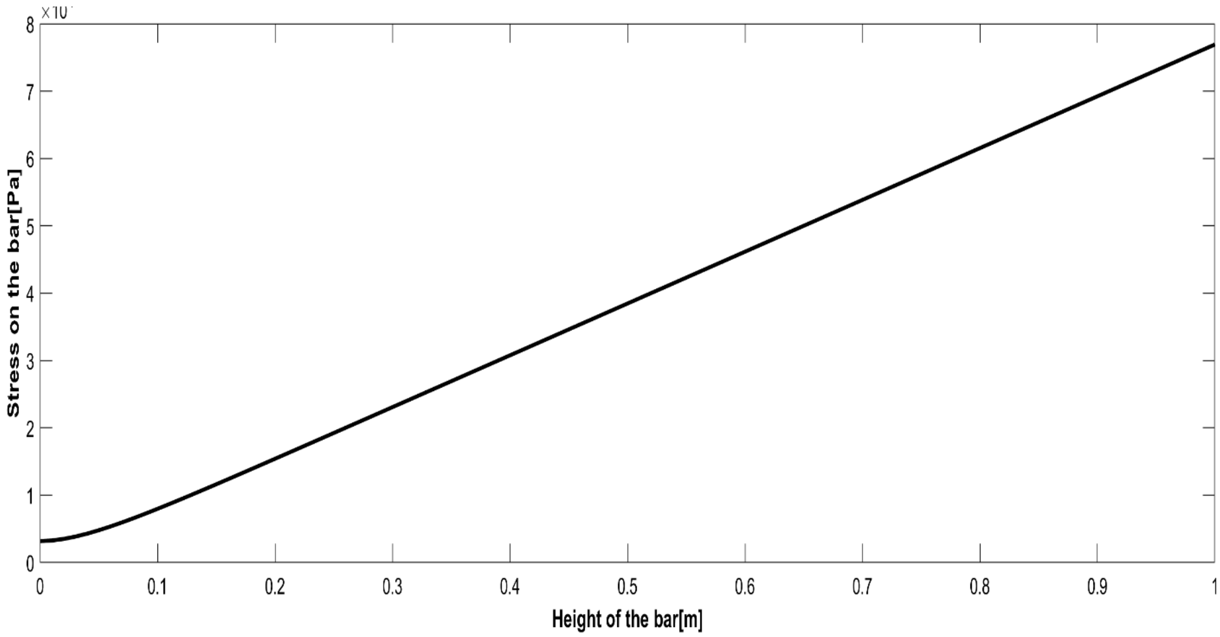

Fig.4: Stress profile

It can be noticed that the height increase causes higher values of normal stresses in the cylinder surface. The weight increase, according to the eq.(13), because of the high density of the steel of 7850[kg/m^3] has much higher impact on the stress value than the area increase.

Instead of exponential function in eq. (11), the unknown generatrix is approximated by **4th degree polynomial**:

$$r(x) = p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5. \tag{14}$$

The coefficients that need to be optimized are obtained as:

p_1= 1.529*10^{-11};

p_2= 1.521*10^{-09};

p_3= 8.854*10^{-07};

p_4= 8.456*10^{-05};

p_5= 0.005.

The 95 % confidence bounds on the fitted coefficients indicate that they are acceptably precise (*Fig.5*). It is shown that the 4th degree polynomials give the best approximation because the generated data follows a polynomial curve.Using higher order polynomials would not be applicable while with the 5th degree polynomial function Matlab indicates that the equation is badly conditioned.

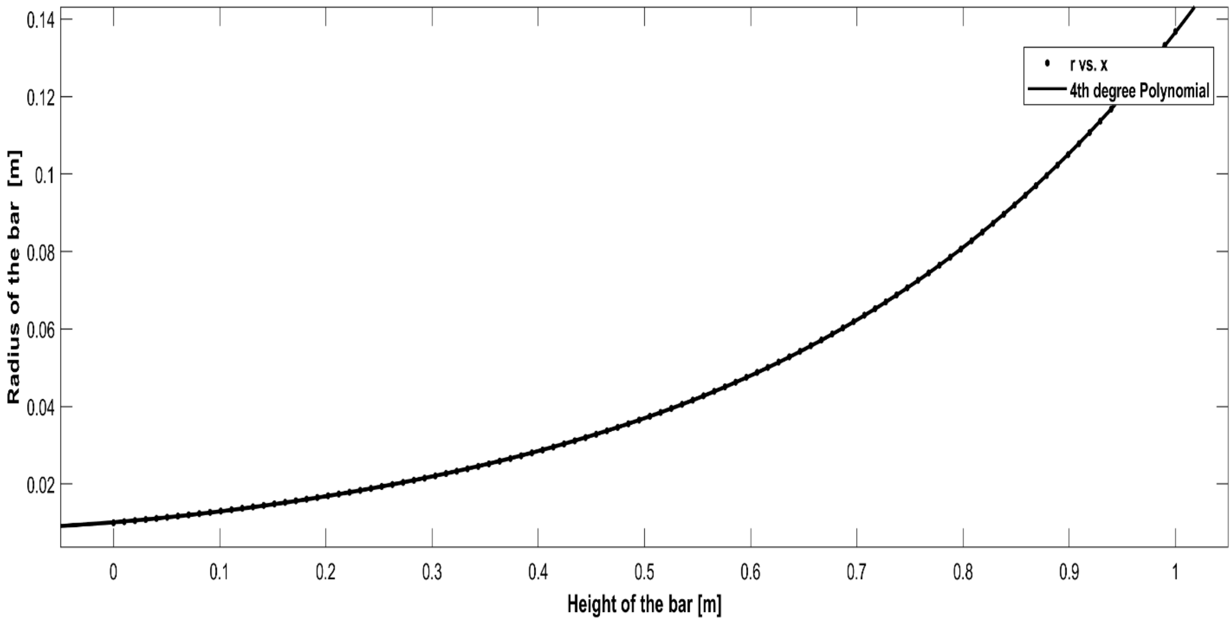

Fig.5: Approximation of the exponential function by 4^th^degree polynomials

4. Weight optimization based on a Least Squares algorithm

The optimization of the shape of the bar, subjected to a compressive load is considered. The problem is reduced to the determination of the minimum weight. Since the developed model is a nonlinear constrained problem, the nonlinear Least-Squares algorithm LSQNONLIN is used.

- **The weight of the bar is taken as the objective function;**
- **A stress constraint is introduced;**
- **The optimization parameters are the coefficients of the polynomial generatrix.**

4.1 Objective function

The objective function is the *total weight of the bar* (as in eq.2):

$$W=W(r) = \int_0^h \rho g r^2 dx \ \rightarrow min. \tag{15}$$

In terms of the 4th degree polynomial function, it can be written as:

$$\widetilde{W} = \rho g \pi \int_0^h (p_1 x^4 + p_2 x^3 + p_3 x^2 + p_4 x + p_5)^2 dx. \tag{16}$$

After integration we obtain:

$$\widetilde{W}(p_1, \dots p_5) = \rho g \pi \left({p_1}^2 \frac{h^9}{9} + {p_2}^2 \frac{h^7}{7} + {p_3}^2 \frac{h^5}{5} + {p_4}^2 \frac{h^3}{3} + {p_5}^2 h \right). \tag{17}$$

4.2 *Constraints*

One of the simplest failure criteria is that the stresses in the structure do not exceed the yield stress of the material. The constraint function is defined in a separate script and the constraint function used for the proposed minimization problem is the stress function. If the admissible stress is denoted as σ_{allow} the constraint function can be written as:

$$Residual = \sigma - \sigma_{allow} = \frac{F+W}{A} - \sigma_{max} = 0, \tag{18}$$

where $\sigma_{max} = 0.9\sigma_{allow}$ (we considered the case with a safety factor).

4.3 *Optimization parameter*

The function coefficients ($p_1, \dots p_5 = optimat$) present the model parameter which need to be optimized. To implement and to solve the proposed problem in Matlab with LSQNONLIN optimization algorithm (in fact minimization problem) it is required that the user specifies lower and upper boundaries for the optimization parameters (coefficients), so that solution is always in the range:

$$lb \leq p \leq ub.$$

Then we set:

$lb = 0$

$ub = 1.d + 29.$

4.4 *Nonlinear Least Squares Algorithm*

A Least Squares method in general, is a problem of finding a vector x that is a local minimizer to a function that is a sum of squares, possibly subject to some constraints [3]:

$$\min_x f(x) = \sum_{i=1}^{m} f_i(x)^2, \tag{19}$$

where the objective function is defined in terms of auxiliary functions f_i with optional lower and upper bounds *lb* and *ub* on the components of x .

$x = lsqnonlin(fun, x_0)$ starts at the point x_0 and finds a minimum of the sum of squares of the functions described in *fun*. The function *fun* should return a vector (or array) of values and not the sum of squares of the values.

In our case:

- $m = 1$;
- $f_1 = \widetilde{W}(p_1, \dots p_5)$;
- $x = (p_1, \dots p_5)$.

The goal of optimization is to determine the model parameters such that the maximum stress in the structure under the designated external force lies within the specified strength limit.
The Least-Squares algorithm tries to approximate the optimum values of model parameter by equalizing the maximum stress in the structure with the predefined admissible stress (eq. 18).
Input values from Matlab:

- *initial radius* r_0*=0.005 [m];*
- *maximum admissible stress:*σ_{allow}*= 7.86 [MPa]-initial stress on the bar taken as maximum admissible stress on the bar;*
- *compressive force on the bar: F=350 [N];*
- *density of the steel:* ρ*= 7850 [kg/*m^3*].*
- *x=100 [m]- length of the bar*

The output structure from Matlab:

- *output = struct with fields;*
- *iterations: 2000;*
- *func-count: 10005;*
- *Residual: 0.463264;*
- *first order optimality: 1.81524e-14;*
- *norm of steps: 3.24.*

The output structure from Matlab shows that algorithm works properly because *Residual* is ≈0. The following optimized coefficients are obtained:

- p_1= $9.41*10^{-12}$;
- p_2 = $9.36*10^{-10}$;
- p_3=$5.45*10^{-07}$;
- p_4=$5.21*10^{-05}$;
- p_5=0.0034.

Starting from the initial cross-sectional dimension, the optimal (minimal) weight W_{opt} is calculated for both representations and the results including the saved material are shown in Table 1.

Tab. 1: Optimized values and saved mass

Representation	W_{init}[N]	W_{opt} [N]	Saved mass [%]
Exponential	3390.86	1116.07	67.08
4^{th} degree Polynomial	3390.33	1112.31	67.19

It can be seen that in the case of the exponential representation the mass is reduced by 67.08 %, and in the case of 4-degree polynomial representation by 67.19 %.

5. Conclusions

This paper proposed a weight optimization method for the uniaxial bar that uses the weight of the bar as the optimization objective. Furthermore, the function coefficients were used as design variables, and the maximum stress as the inequality constraint.

A Least-Squares algorithm (LSQNONLIN) was used as the optimization algorithm.

The maximum stress depends on the loading conditions. It is also possible to perform the proposed optimization for different loading cases (torsion, tension etc.), and for other approximations of the generatrix, e.g. by cubic splines.

In the same way it would be possible to calculate the saved mass when the maximum displacements or strain energy are taken as the inequality constraints.

The conclusions in this paper are as follows:

1) The weight optimization method for the uniaxial bar is correct and efficient.
2) The structural mass decreased substantially after optimization.
3) The user has to specify the stress limiting value for the particular situation. In this paper, a initial stress on the bar of 7.86 [MPa] was taken as maximum admissible stress.0.

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